Equivalence of impact phase models in two vehicle planar collisions

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Abstract
Determining the pre-impact velocities of vehicles are of prime importance when investigating road traffic collisions. Two types of impact-phase model are in common use to achieve this purpose, those based on the conservation of linear and angular momentum as exemplified by the models presented by Brach and Ishikawa and the CRASH model which explicitly includes the conservation of energy. A summary of the various models is provided to show how the models are related to each other together with a brief discussion of their strengths and weaknesses. Of particular significance is that although there are differences between these models it is shown that they are equivalent provided certain conditions are satisfied, namely that the crush or impact plane is orientated perpendicular to the impulse. In addition it is shown that they produce identical results from consistent input data. Explicit conversion factors between the models are provided together with a novel method to transform coefficients of restitution between various orientations of the crush plane. This facilitates comparison and movement between the models and it is shown that the choice of model utilised for an individual collision depends largely on the availability of particular data.

Keywords: speed change, velocity change, momentum, CRASH, vehicle collisions

1. Introduction

From the perspective of a forensic investigator a collision between two vehicles can be considered as comprising three main phases. There is an initial pre-impact phase where the vehicles move towards impact, a collision phase where the vehicles interact with each other and a post-impact phase where the motion of the vehicles from impact towards rest is considered. The pre and post impact phases are concerned mainly with the analysis of tyre and other marks on the road surface. Techniques to establish the speeds of vehicles from these marks are well established e.g. Neades [**1**] and other simple techniques are described by Lambourn [**2**] and Neades [**3**]. The presence of water on a road surface and the increase in ABS braking systems decreases the chance of suitable tyre marks being found on the road surface and there are a variety of methods that provide information on vehicle speeds in the absence of tyre marks. One such method involves the use of the pedestrian throw distance discussed, for example, by Evans and Smith [**4**].

Where there are insufficient tyre marks, an analysis of the impact phase of the collision is often the only source of information concerning the behaviour and speeds of the vehicles. Impact phase models tend to fall into two broad categories, those based solely on the conservation of linear and/or angular momentum and the CRASH model which explicitly includes the conservation of energy. Two commonly used momentum based models are described by Brach [**5**] and Ishikawa [**6**]. A third commonly used model is the CRASH algorithm and Smith [**7**] describes how this model can be derived solely from conservation laws.

An important extension to the impact phase models is demonstrated by Neades and Smith [**8**] where they discuss how an analysis of the change in velocity can be used to determine the actual velocities of vehicles involved in a collision. Their model can be used with DeltaV data obtained either from any of the crash phase models or directly from vehicle data recorders.

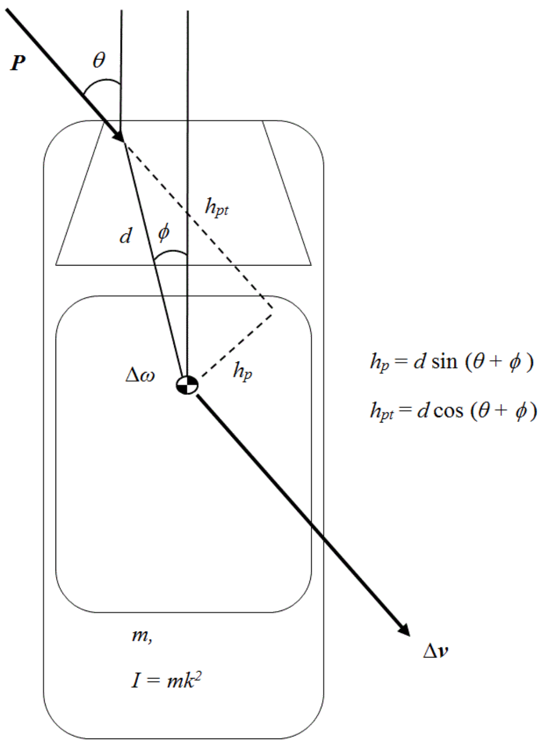
A summary of the various models is provided in the next section to show how the models are related to each other together with a brief discussion of their strengths and weaknesses. Although there are differences between these models it is shown that they are equivalent and that they produce identical results from consistent input data. Explicit conversion factors between the models are provided to facilitate comparison and movement between the models. It is also shown that the choice of model utilised for an individual collision depends largely on the availability of particular data.

2. Common theory and assumptions

Impact phase models commonly make a number of assumptions and are restricted to the analysis of two vehicle planar collisions. First tyre and other external forces are assumed to be negligible during the impact, so that momentum is conserved. Second, the vehicle masses and moments of inertia are maintained throughout the collision. That is, the deformations caused by the collision do not significantly change the moments of inertia and the masses of the vehicles are not significantly changed, for example, by parts of a vehicle becoming detached as a result of the collision. Third, the time-dependent impulse can be modelled by one force, its resultant (***P***), which acts at some point on or in the vehicles to cause a change in both linear and angular velocity ($\Delta\boldsymbol{v}$ and $\Delta\omega$ respectively).

A diagram showing a vehicle based reference frame is shown in Figure 1. The position of the point of application of the impulse relative to the centre of mass of a vehicle can be described using the distance d and angle ϕ. The parameter h_p is the length of the moment arm of the impulse about the centre of mass and is dependent on the position of the point of application of the impulse and the principal direction of force θ. The length of the moment arm tangential to the impulse h_{pt} is also relevant to these models and is discussed later.

Figure 1: Vehicle based reference frame



The conservation of linear momentum leads to the equations

$$m_1(\boldsymbol{v}_1 - \boldsymbol{u}_1) = m_1\Delta\boldsymbol{v}_1 = \boldsymbol{P}, \tag{1}$$
$$m_2(\boldsymbol{v}_2 - \boldsymbol{u}_2) = m_2\Delta\boldsymbol{v}_2 = -\boldsymbol{P} \tag{2}$$

where m is the mass of each vehicle, $\boldsymbol{P}$ is the impulse and $\boldsymbol{u}$ and $\boldsymbol{v}$ are the initial and final velocities and $\Delta\boldsymbol{v}$ is defined as the change in velocity $\boldsymbol{v}$ - $\boldsymbol{u}$. Subscripts 1 and 2 refer throughout to vehicles 1 and 2 respectively. In collinear collisions, the line of action of the impulse $\boldsymbol{P}$ passes through the centres of mass of the vehicles and there is no change in the rotational velocity of either vehicle. If $\boldsymbol{P}$ does not act through the centres of mass it produces a change not only in the motion of the centres of mass, but also a rotation of each vehicle about the centre of mass given by

$$m_1k_1^2(\Omega_1 - \omega_1) = m_1k_1^2\Delta\omega_1 = h_{1p}P, \tag{3}$$
$$m_2k_2^2(\Omega_2 - \omega_2) = m_2k_2^2\Delta\omega_2 = -h_{2p}P \tag{4}$$

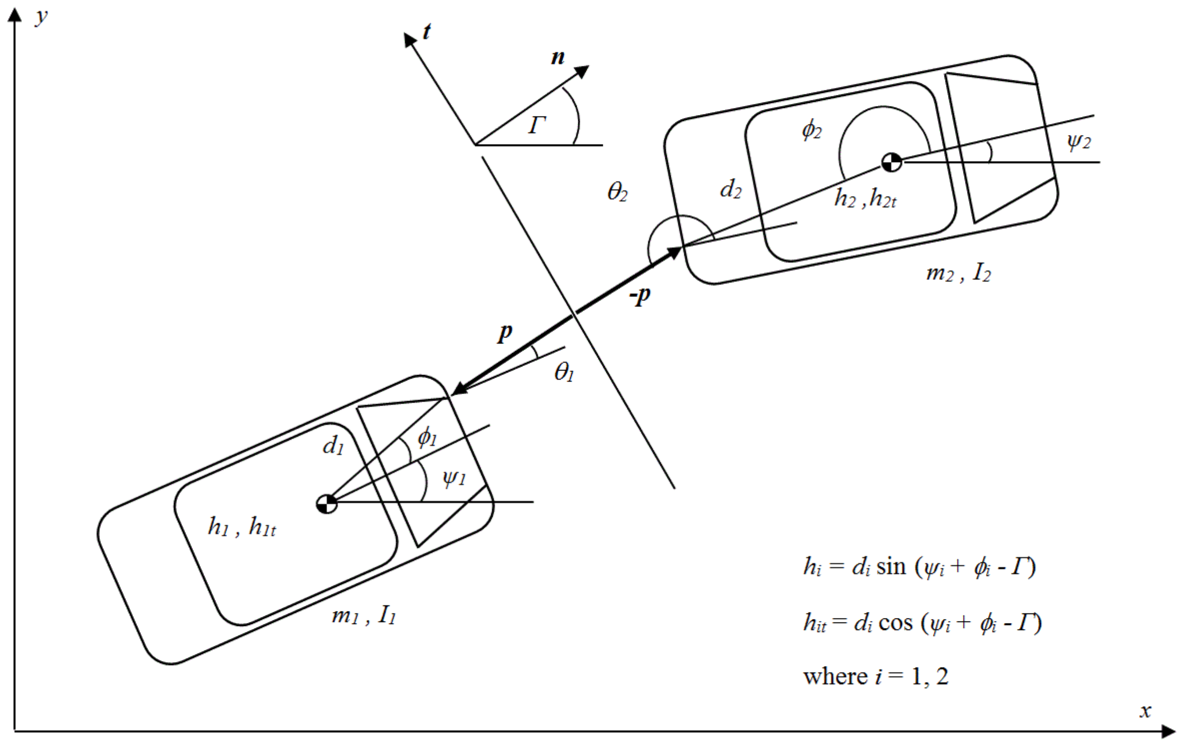
where k is the radius of gyration, h_p the moment arm of the impulse about the centre of mass, ω and Ω are the pre and post-impact rotational velocities of each vehicle and $\Delta\omega$ represents the change in rotational velocity $\Omega - \omega$. In vehicle to vehicle collision it is not unreasonable to assume that the masses, radii of gyration and moment arms for each vehicle are known or can be obtained easily. Equations (1) to (4) then form a system of four equations with eight unknown velocity variables. Provided that four velocity variables can be established, or other constraining equations such as restitution or energy are added, then complete solutions for the remaining four variables can be determined. The momentum based models of Brach [**5**] and Ishikawa [**6**] utilise these equations and attempt to provide methods to establish solutions for the unknown velocities and these are described in the next section.

3. Momentum-only models

Brach [**5**] describes a Planar Impact Mechanics (PIM) model where he considers the conservation of linear and angular momentum in a orthonormal coordinate system oriented to an impact or crush plane which is established parallel to a hypothetical contact surface common to both vehicles. The crush plane is related to the *x*-*y* coordinate system by the angle Γ as shown in Figure 2. The parameter Γ essentially defines the normal $\boldsymbol{n}$ to the crush plane. Subscripts n and t are used to represent component variables normal and tangential to the crush or impact plane and coincide with the orientation of the unit vectors $\boldsymbol{n}$ and $\boldsymbol{t}$ shown Figure 2.

The impulse $\boldsymbol{P}$ due to impact is resolved into two components, normal and tangential to the crush plane. The resulting six equations and eight unknowns are supplemented with two coefficients to provide additional constraints and thereby generate a solution. Brach defines a coefficient of restitution normal to the crush plane (e_n) which is defined as the ratio of the relative normal velocity post impact to the relative normal velocity pre impact. Brach also introduces another coefficient, the impulse ratio μ. This is effectively a coefficient of friction and is defined as the ratio of the normal and tangential impulse components. Brach's solution to equations (1) to (4) are a series of equations which are shown in Appendix B. From these equations it is straightforward to determine the total change in velocity ($\Delta\boldsymbol{v}$) of each vehicle.

Figure 2: Collision configuration



Brach [**9**] shows that an important quantity in this model is the value μ_0 which is the impulse ratio μ that provides a common post-impact velocity (***V***) tangential to the crush plane, i.e. where $V_{1t} = V_{2t}$. For vehicle to vehicle collisions the point of application of the impulse on each of the vehicles frequently reach a common velocity. The common velocity condition is satisfied in this model when e_n = 0 and $\mu = \mu_0$. The parameter μ_0 is the ratio required to achieve a common velocity tangentially to the impact plane and is described by Brach as the critical impulse ratio such that

$$\mu_0 = \frac{rA + B(1+e_n)}{(1+e_n)(1+C) + rB} \qquad (5)$$

where r is the ratio of the closing velocity components perpendicular and tangential to the crush plane, i.e. $r = U_{Rt} / U_{Rn}$. The coefficients A, B and C are simplifying parameters and are defined in Appendix B.

Ishikawa's model [**6**] is similar in many respects to planar impact mechanics proposed by Brach. Ishikawa also defines a crush plane to resolve the impulse into normal and tangential components. Where Ishikawa's model differs from Brach is that he proposes the utilisation of two coefficients of restitution, one normal to the impact plane (e_n) and the other tangential to the impact plane (e_t). These are defined such that the relative velocities of the point of application before (U) and after (V) impact are given by

$$V_{Rn} = -e_n U_{Rn}, \qquad V_{Rt} = -e_t U_{Rt} \qquad (6)$$

where

$$
\begin{aligned}
U_{Rn} &= u_{2n} - h_2\omega_2 - u_{1n} + h_1\omega_1, \qquad & V_{Rn} &= v_{2n} - h_2\Omega_2 - v_{1n} + h_1\Omega_1, \\
U_{Rt} &= u_{2t} + h_{2t}\omega_2 - u_{1t} - h_{1t}\omega_1, \qquad & V_{Rt} &= v_{2t} + h_{2t}\Omega_2 - v_{1t} - h_{1t}\Omega_1.
\end{aligned} \tag{7}
$$

Ishikawa provides a solution for the impulse components, P_n and P_t using the relative closing speeds and relative separation speeds at impact. Ishikawa's solutions are shown in Appendix C. From the impulse components, P_n and P_t it is straightforward to use equation (1) or (2) to determine the change in velocity sustained by each vehicle. If either the post-impact or pre-impact velocities are known, then it is then possible to determine the remaining linear velocities. Further, the change in rotation can be derived from equations (3) and (4)

Ishikawa also shows that the two coefficients of restitution e_n and e_t are related to the impulse ratio μ used in Brach's model by the equation

$$
e_t = \frac{m_n U_{Rn}(1+e_n)(\mu - m_t m_0)}{m_t U_{Rt}(1-\mu m_n m_0)} - 1. \tag{8}
$$

With the obvious constraint that the same orientation of the impact plane is used in both Ishikawa's and Brach's models and that there is a common value for e_n, equation (8) provides a useful way of converting Brach's impulse ratio μ into Ishikawa's tangential coefficient of restitution e_t. In the reverse scenario, the normal and tangential components determined from Ishikawa's model can be used to define Brach's impulse ratio

$$
\mu = \frac{P_t}{P_n} = \frac{m_n m_t m_0 U_{Rn}(1+e_n) + m_t U_{Rt}(1+e_t)}{m_n m_t m_0 U_{Rt}(1+e_t) + m_n U_{Rn}(1+e_n)}. \tag{9}
$$

This can be solved for the normal coefficient of restitution e_n in terms of the tangential coefficient e_t and the ratio of the impulse components μ to give equation (8). Equation (9) can also be solved to give an expression relating the two coefficients of restitution i.e.

$$
\frac{1+e_t}{1+e_n} = \frac{m_n U_{Rn}(\mu - m_t m_0)}{m_t U_{Rt}(1-\mu m_n m_0)}. \tag{10}
$$

There are clear similarities between the coefficients A, B and C used in Brach's model and the coefficients m_n, m_t and m_0 used in Ishikawa's model. Analysis shows that the coefficients are related by the expressions

$$
\bar{m} = \frac{m_1 m_2}{(m_1 + m_2)}, \qquad m_n = \frac{\bar{m}}{A}, \qquad m_t = \frac{\bar{m}}{(1+C)}, \qquad m_0 = \frac{B}{\bar{m}}. \tag{11}
$$

Appendix D shows these relationships and derived products which facilitate conversion between Brach's model and that of Ishikawa. Substitution of equations (11) into equation (10) and solving for μ produces the expression

$$
\mu = \frac{(1+e_t)rA + B(1+e_n)}{(1+e_n)(1+C) + rB(1+e_t)}. \tag{12}
$$

Equation (12) demonstrates how the restitution coefficients used by Ishikawa are related to the *A*, *B* and *C* parameters used by Brach. In particular it is noted that when e_t is zero then equation (12) simplifies to become identical to Brach's critical impulse ratio shown in equation (5).

4. The CRASH model

Although originally intended as a tool for assessing accident severity by McHenry [**10**], CRASH has been widely adopted by the crash investigation community. Where there is insufficient information as to the desired output velocities the momentum-only models cannot succeed. Information about the collision severity and changes in velocity can still be obtained from an analysis of the damage sustained by each of the vehicles and this is the basis for CRASH. The model utilises the conservation laws of momentum and energy to establish the change in velocity (Δv) of a vehicle from the damage sustained by each vehicle (E_1 and E_2). The assumption is made that the points of application of the impulse reach a common velocity during the approach phase of the collision. With this assumption, Tsongas [**11**] shows that the CRASH equation can be expressed as

$$\Delta v_1 = \sqrt{\frac{2\gamma_1\gamma_2 m_2(E_1+E_2)}{m_1(\gamma_1 m_1+\gamma_2 m_2)}} = \sqrt{\frac{2\gamma_1(E_1+E_2)}{m_1\left(1+\dfrac{\gamma_1 m_1}{\gamma_2 m_2}\right)}} \tag{13}$$

where γ is defined in a similar manner as Ishikawa's model [**6**] i.e. as

$$\gamma = \frac{k^2}{k^2+h_p^2}\,. \tag{14}$$

Smith [**7**] shows that some relaxation to the common velocity condition can be achieved by incorporating a coefficient of restitution parallel to the impulse e_p. His derivation provides an expression for the change in velocity

$$\Delta v_1 = \sqrt{\frac{2\gamma_1\gamma_2 m_2(E_1+E_2)(1+e_p)}{m_1(\gamma_1 m_1+\gamma_2 m_2)(1-e_p)}} = \sqrt{\frac{2m_2(E_1+E_2)(1+e_p)}{m_1(m_1\delta_2+m_2\delta_1)(1-e_p)}} \tag{15}$$

where $\delta = 1/\gamma$.

The change in velocity calculated by this method is the change in velocity of the centre of mass of each vehicle along the line of action of the impulse. From Newton's Second Law it follows that there can be no change in velocity at the centre of mass tangential to the impulse so CRASH implicitly defines the total change in velocity at the centre of mass.

Note that equations (13) and (15) can be derived solely from conservation laws and do not define how the crush energy values E_1 and E_2 might be obtained. A variety of methods could be utilised to determine the crush energy; however the description by McHenry [**10**] also included a method for determining these parameters. McHenry's CRASH algorithm may therefore be more accurately viewed as two distinct algorithms, the first to estimate of the work done in causing deformation and the second to use those values to determine the change in velocity. It is not the purpose here to comment on any method used to determine E_1 or E_2 rather it is to demonstrate how the CRASH model compares with the momentum models and this is discussed next.

5. Equivalence of CRASH and the momentum-only models

Brach and Brach [**5**] and [**12**] show how the momentum change in each vehicle can be written using that model as

$$m_i \Delta v_i = \sqrt{\frac{2\bar{m}(1+\mu^2) q E_L (1+e_n)}{(1-e_n) + 2\mu r - \dfrac{\mu r^2}{\mu_c}}} \tag{16}$$

where E_L is the total energy loss in the collision (i.e. $E_1 + E_2$), q and r are as defined in Appendix B and

$$\mu_c = r/(1+e_t)\,. \tag{17}$$

(Note that in Brach and Brach [**12**] equation (16) appears to have been misprinted so that the $(1+\mu^2)$ term appears incorrectly as $(1+\mu)$ and the numerator in the final term reads incorrectly as $\mu^2 r$ instead of μr^2.) It is possible to align the crush plane used in Brach's and Ishikawa's models so that it is perpendicular to the impulse $\boldsymbol{P}$. With this orientation of the crush plane the tangential impulse component must be zero so that the ratio of the tangential and normal components μ is also zero. This leads to a simplification of equation (16) to

$$m_i \Delta v_i = \sqrt{\frac{2\bar{m} q E_L (1+e_n)}{(1-e_n)}}\,. \tag{18}$$

Furthermore, since μ is zero, q in Appendix B can also be simplified and can be found from

$$\frac{1}{q} = 1 + \frac{\bar{m} h_1^2}{m_1 k_1^2} + \frac{\bar{m} h_2^2}{m_2 k_2^2}\,. \tag{19}$$

Equation (19) can be solved for q and substituted into equation (18) to give

$$m_i \Delta v_i = \sqrt{\frac{2 m_1 m_2 E_L (1+e_n)}{\left[m_1 k_1^2 (k_2^2 + h_2^2) + m_2 k_2^2 (k_1^2 + h_1^2)\right](1-e_n)}}\,. \tag{20}$$

The CRASH solution as shown in equation (15) can also be written in a similar manner to equation (18) to show the change in momentum of each vehicle, i.e.

$$m_i \Delta v_i = \sqrt{\frac{2\gamma_1 \gamma_2 m_1 m_2 (E_1 + E_2)(1+e_\mathrm{p})}{(\gamma_1 m_1 + \gamma_2 m_2)(1-e_\mathrm{p})}}\,. \tag{21}$$

Substituting for γ as defined in equation (14), equation (21) can be expanded to produce

$$m_i \Delta v_i = \sqrt{\frac{2m_1 m_2 (E_1 + E_2)(1+e_p)}{\left[m_1 k_1^2 (k_2^2 + h_{2p}^2) + m_2 k_2^2 (k_1^2 + h_{1p}^2)\right](1-e_p)}}. \tag{22}$$

In general the crush plane required by the momentum-only models of Brach and Ishikawa is not perpendicular to the impulse $\boldsymbol{P}$. As a result the length of the moment arm h_p used by the CRASH model is not equal to the length of the moment arm h used in Brach's and Ishikawa's models. In CRASH the moment arms h_p and h_{pt} are defined relative to the impulse $\boldsymbol{P}$ but in both Brach and Ishikawa's models the moment arms h and h_t are defined relative to the crush plane. However if the crush plane is orientated so that it is perpendicular to $\boldsymbol{P}$ it can be seen that equation (22) is equivalent to equation (20) with $h_p = h$ for each vehicle, $e_p = e_n$ and $E_L = E_1 + E_2$. It can also be seen that although CRASH does not require an impact plane to be defined, it can be used to define a crush plane; one which is perpendicular to the impulse $\boldsymbol{P}$.

CRASH implicitly assumes that a common velocity is achieved tangentially at the point of application of the impulse, i.e. $V_{1t} = V_{2t}$. In Ishikawa's model this is achieved when $e_t = 0$. In Brach's model a common tangential velocity is achieved when $\mu = \mu_0$. Neades and Smith [**8**] demonstrate that there can be no change in velocity tangentially to the impulse at the centre of mass. Any change in velocity of the points of action tangential to the impulse can therefore only be due to a change in the angular velocity of the vehicle, so that

$$\Delta V_{1t} = h_{1pt}\ \Delta\omega_1, \qquad \Delta V_{2t} = h_{2pt}\ \Delta\omega_2. \tag{23}$$

This summary shows that all three models are different formulations of solid body collisions. Brach's model utilises the conservation of momentum, restitution and friction and requires the definition of a crush plane which may prove difficult to specify for a real collision. Ishikawa's model is essentially the same as Brach's model with the replacement of the friction parameter with a tangential coefficient of restitution. CRASH however utilises the conservation of momentum and energy and requires the direction of the impulse to be defined. In the next section a method is developed whereby non-zero coefficients e_n, μ (or e_t) can be transformed between different orientations of the crush plane. This facilitates the conversion between the momentum-only models and CRASH.

6. Transforming restitution coefficients

Brach, Welsh and Brach [**13**] identify that the orientation of the impact plane is immaterial in collisions where there is a common post-impact velocity (i.e. $e_n = 0$ and $\mu = \mu_0$ or $e_t = 0$). However in situations where there is some relative motion after impact, either normal or tangential to the crush plane, then the choice of orientation becomes relevant as the results are then dependent upon the values chosen for the coefficients e_n, μ (or e_t). It follows therefore that where there is relative motion after impact, then in order to maintain consistent results with alternative orientations of the crush plane, the values of e_n, μ (or e_t) will need adjustment. The relationship between the impulse components and the partial work done by those components provides a way of determining the relative values of those coefficients. This is discussed in detail by Neades [**14**] and summarised below.

A result first noted by Kelvin and Tait [**15**] and expanded by Stronge [**16**] enables the total work in a collision to be partitioned into normal and tangential terms. Using the subscript *i* for each term, their results state that the partial work (W_i) done on colliding bodies by the component of the reaction impulse (P_i) equals the scalar product of this component and half the sum of the initial (U_i) and final (V_i) velocities of the contact point in the direction of this impulse component i.e.

$$W_i = \frac{P_i}{2}(U_i + V_i)\,. \tag{24}$$

In a planar collision the total work done is the sum of the work done by the normal and tangential components of the impulse. Also, the initial (U_i) and final (V_i) velocities are more usefully described in terms of their relative velocity components so that the total work done in a collision can be described using equation (24) as

$$W = \frac{P_n}{2}[(U_{2n} - U_{1n}) + (V_{2n} - V_{1n})] + \frac{P_t}{2}[(U_{2t} - U_{1t}) + (V_{2t} - V_{1t})] \tag{25}$$

Together with the definitions of e_n and e_t and U_{Rn}, U_{Rt}, V_{Rn}, and V_{Rt} as defined by Ishikawa [**6**] and shown in Appendix C, this allows the total work done in a collision to be expressed as

$$E_L = \frac{P_n U_{Rn}}{2}(1 - e_n) + \frac{P_t U_{Rt}}{2}(1 - e_t). \tag{26}$$

Brach [**9**] shows that the components of an impulse normal and tangential to a crush plane can be calculated from the equations

$$P_n = \frac{\bar{m}(1 + e_n)U_{Rn}}{A - \mu B} \tag{27}$$

$$P_t = \mu P_n. \tag{28}$$

The magnitude of the total impulse is the vector sum of the normal and tangential components so that

$$P^2 = P_n^2 + P_t^2 = P_n^2\left(1 + \mu^2\right). \tag{29}$$

Brach also defines a parameter *r* such that $r = U_{Rt}/U_{Rn}$ which together with equations (27) to (29) can be substituted into equation (26) and subsequently solved for *P* to give

$$P = \sqrt{\frac{2E_L\bar{m}\left(1 + e_n\right)\left(1 + \mu^2\right)}{\left(A - \mu B\right)\left[1 - e_n + \mu r\left(1 - e_t\right)\right]}}. \tag{30}$$

Of note is that equation (30) is effectively the same as equation (16) with the substitution of $q = 1/\left(A - \mu B\right)$ and $\mu_c = r/(1 + e_t)$.

Equation (12) expresses the relationship between Brach's impulse ratio μ and Ishikawa's two restitution coefficients e_n and e_t and can be written as

$$(1-e_t)=2-\frac{(1+e_n)[\mu(1+C)-B]}{r(A-\mu B)}. \tag{31}$$

Equation (31) can then be substituted into equation (30) to eliminate e_t and the resultant equation solved for e_n to give

$$e_n=\frac{[R(1+2\mu r)-S-T]}{R+S+T} \tag{32}$$

where

$$\begin{aligned} R &= A-\mu B, \\ S &= \mu^2\left(1+C\right)-\mu B, \\ T &= \frac{2E_L\bar{m}}{P^2}\left(1+\mu^2\right). \end{aligned} \tag{33}$$

For any planar collision, once E_L, the total work done in causing crush and P, the total impulse, are established for one particular orientation of the crush plane, then these totals must apply to every other orientation of the crush plane if the same output results are to be maintained. As the crush plane is rotated about the impulse, the values of A, B and C change as the proportion of normal and tangential components varies. These parameters are essentially geometric and as such are dependent on the orientation of the crush plane Γ. For a collision, the value of μ can be defined as the tangent of the angle of the impulse. A value for r can be defined similarly as the tangent of the angle of the closing speed vector. The difference between these angles (angle ζ) will remain constant whatever the orientation of the crush plane, i.e. $\zeta=\tan^{-1}(r)-\tan^{-1}(\mu)$. Equation (32) can therefore be utilised to find the value of e_n for any other orientation and the equivalent tangential coefficient of restitution e_t can be found from equation (31).

The case when $\mu=0$ is of particular interest as this corresponds to the orientation of the crush plane required to align with CRASH. As indicated previously, in order to maintain equivalence, the normal coefficient of restitution required by each of the models must be the same, i.e. $e_n = e_p$. With this orientation the lengths of the moment arms h_p and h_{pt} match the corresponding moment arms h and h_t used in the momentum-only models. When $\mu=0$, the numerator in Equation (12) is equal to zero, *i.e.*

$$0=rA\left(1+e_t\right)+B\left(1+e_n\right). \tag{34}$$

It is clear that a non-zero tangential coefficient of restitution may be required to maintain consistent results between the models. This necessitates some modification to the closing speed algorithm developed by Neades and Smith [**8**] through the incorporation of a tangential coefficient of restitution. Neades [**14**] describes how this may be achieved by substituting the definition of tangential restitution given by equation (6) into equation (23) to give

$$U_{2t}-U_{1t}=(h_{1pt}\ \Delta\omega_1-h_{2pt}\ \Delta\omega_2)/(1+e_t). \tag{35}$$

This modifies the calculated tangential closing velocity component and the ratio of the tangential and normal closing velocity components required by that model. In that model this ratio is expressed as $r = \tan\beta$ Furthermore, since the lengths of the moment arms are also equivalent ($h_p = h,\ h_{pt} = h_t$), this ratio can be expressed more conveniently using equation (34) *i.e.*

$$\tan\beta = r = -\frac{B(1+e_n)}{A(1+e_t)}. \qquad (36)$$

7. Discussion

In the discussion earlier it is seen that the three models considered use the same conservation laws of linear and angular momentum and energy either explicitly or implicitly. As such it should not perhaps be surprising that they are equivalent. Despite the similarities between the models there are differences. Both the momentum models are designed to be used in an iterative manner by adjusting the input velocities to produce the desired output velocities. CRASH is designed to take as input the energy loss due to deformation and the direction in which the impulse acts on the vehicle from which it produces a change in velocity.

The direction is which the impulse acts is commonly known as the principal direction of force (PDOF) The requirement to estimate a PDOF is a regarded as a major weakness by several commentators (e.g. Brach [**9**], Woolley [**17**]) since it is difficult to estimate this quantity reliably or consistently. Smith and Noga [**18**] for example suggest that the PDOF for each vehicle may be subject to a range of ±20° for different investigators. To alleviate this problem Neades and Smith [**8**] describe a technique for refining an initial estimate of the PDOF. Essentially this algorithm utilises change in velocity data to generate pre- and post-impact velocities. This information can then be used to refine the orientation of the impulse so that the output directions of travel match those of the actual vehicles involved. In effect, that technique uses a combination of both the CRASH and momentum models. A similar approach is also suggested by Brach, Welsh and Brach [**13**].

In an analogous manner to the choice of PDOF, there is some freedom to define the orientation of the crush plane required by the momentum models and this parameter requires an estimate by the investigator. The choice of orientation of the impact plane is discussed in detail by Brach, Welsh and Brach [**13**] and they indicate that a nominal crush plane for a collision is one that is orientated so that it bisects the angle between the contacting surfaces of the two vehicles at impact. Earlier work by Brach [**12**] and Ishikawa [**19**] suggest orientation to an axis parallel to a hypothetical flat crush plane common to both vehicles. It is suggested that such choices are likely to align closely with the perpendicular orientation to the impulse required to match with CRASH as outlined previously. The requirement to specify a crush plane in the momentum-only models presents similar practical difficulties to an investigator as those encountered when specifying the PDOF.

The use of consistent data sets for each of the models generates identical results. For example it is possible to use energy loss calculated using the momentum models as input to the CRASH model. Similarly the pre-impact velocities calculated using CRASH together with the model developed by Neades and Smith [**8**] and extended here, can be used as input to the momentum models. Both scenarios produce identical results from each of the models and an example collision is discussed in the next section.

This important result indicates that where differences do exist when dealing with practical scenarios, they are likely to be due to differences and inconsistencies between the sets of input data. All three models require a choice of values for a variety of parameters such as e_n e_t μ γ or Γ. For example a variety of methods could be used to determine the energy loss required as input to the CRASH algorithm and it is difficult to determine these values to a high level of accuracy. Similarly it is difficult to specify the values of e_n or μ_0 for a particular collision. In many cases too, post-impact velocities are not known to a high level of accuracy which reduces the utility of the momentum models. In practice the availability and accuracy of the source data is one factor that helps to determine the choice of model utilised by an investigator. For example consider a simple collinear head-on collision between two vehicles where the vehicles stop as a result of the impact. The momentum-only models cannot provide a solution as to the pre-impact speed of either vehicle. However the CRASH model could be used to establish the change in velocity of each vehicle and thereby the pre-impact velocities using the model developed by Neades and Smith [**8**].

The use of particular data sets to obtain a solution for each of the models is highlighted in the next section where an example collision is used to illustrate how each of the models can be used to generate consistent results.

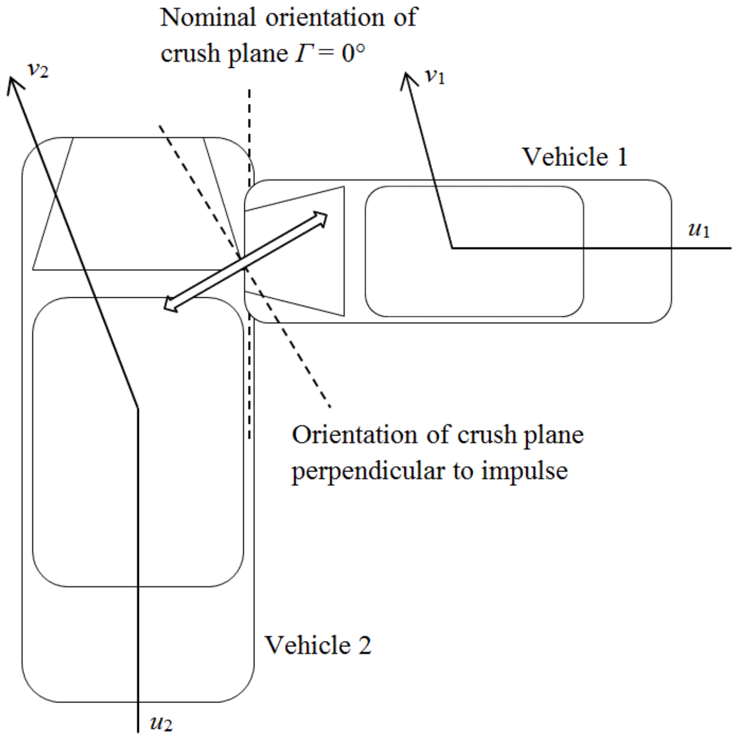
8. Example collision

Validation of each of the models is provided individually by the authors of each of the models and the main purpose of this discussion is to demonstrate the equivalence of those models. To demonstrate how the data from an actual collision may be used as input to each of the models, a collision was selected from the Research Input for the Computer Simulation of Automobile Collisions full scale tests (RICSAC) [**20**]. Several authors have analysed these tests and a number of discrepancies in the analyses are apparent, e.g. Smith and Noga [**21**], and Brach and Brach [**9**]. For the purpose of this discussion however these differences are unimportant.

For consistency the data reported by Brach and Brach [**9**] for Test 9 is utilised as this provides the most complete description of the input data used for his model. Test 9 of the series was set up to be representative of a 90° intersection collision with both vehicles travelling at 9.48 ms^{-1} at impact. The impact configuration is shown in Figure 3 with the available data shown in Table 1 in Appendix E.

Normally the pre-impact velocities are unknown and are obtained by iteration using Brach's model until the desired output velocities are achieved. For this collision Brach and Brach's solution [**9**] utilises the actual pre-impact velocities as input. The orientation of the crush plane Γ used was zero. An additional constraint for this solution was imposed by assuming that there was no tangential post impact motion at the point of application of the impulse, i.e. $\mu = \mu_0$. This corresponds to $e_t = 0$ using Ishikawa's model [**6**]. Their result was optimised to find the coefficient of restitution e_n that most closely matched the post-impact velocities as determined from analyses of the accelerometers fitted to each vehicle as described by Brach [**22**]. They found that this was achieved with $e_n = 0.355$ which produced a value for μ_0 of 0.512. The results are summarised in Table 2 in Appendix E.

Figure 3: Impact configuration for RICSAC 9



The energy loss calculated for this scenario was 42.7 kJ with a total impulse of 8750 Ns Orientating the crush plane so that it is perpendicular to the impulse, i.e. $\Gamma = 27.1°$ allows the corresponding coefficients of restitution to be calculated using equations (32) and (31). It is found that with this orientation $e_n = 0.235$ and $e_t = -0.373$. Using the energy loss of 42.7 kJ and $e_p = e_n$ as input to CRASH (equation (15)) produces identical changes in velocity data. The change in velocity data can also be utilised in the Neades and Smith algorithm [**8**] together with the two coefficients of restitution to produce the same pre- and post-impact velocities as determined using Brach's model.

CRASH requires the PDOF for each vehicle and work done in causing crush as input. One method for obtaining estimates of the work done in causing crush is described by McHenry [**10**] and the measurement process is detailed by Neades and Shephard [**23**]. For this collision and by applying the adjustments to the measurements suggested by Neades and Smith [**8**] it is found that the work done in causing crush was 20.4 kJ for vehicle 1 and 10.2 kJ for vehicle 2. The PDOF values used are as estimated by Jones and Baum [**20**] and shown in Table 2. Assuming a zero coefficient of restitution $e_p = 0$ and using equation (15) generates change in velocity data of 5.65 ms^{-1} and 2.60 ms^{-1} for vehicles 1 and 2 respectively.

In isolation CRASH generates only the changes in velocity of the centre of mass of each vehicle. The model of Neades and Smith [**8**] can be used with the change in velocity data to determine pre- and post-impact velocities. In addition their model allows the PDOF values to be refined so that the calculated post-impact directions of travel match those found from an analysis of the collision scene. It is found that a coefficient of restitution $e_p = 0.3$ the calculated post-impact directions of travel match well with the measured values. Using this coefficient of restitution equation (15) generates changes in velocity of 7.69 ms^{-1} and 3.54 ms^{-1} for vehicles 1 and 2 respectively. The pre-impact speeds are then found using the Neades and Smith model to be 8.18 ms^{-1} for vehicle 1 and 8.03 ms^{-1} for vehicle 2. Using these speeds in the Brach and Brach model generates the same post-impact velocities and estimation of work performed. These results are summarised in Table 2 in Appendix E.

9. Conclusions

It has been shown that the two commonly used momentum based models of Brach [**5**] and Ishikawa [**6**] are essentially different representations of the same model. They utilise different parameters and explicit conversion factors between the two sets of parameters are provided in Appendix D. The analysis presented here facilitates an easy comparison between all the models and provides a novel method to transform coefficients of restitution between different orientations of the crush plane.

It is found that if the crush plane required by the momentum-only models is aligned so that it is orientated perpendicular to the impulse $\boldsymbol{P}$, then the CRASH model is equivalent to the momentum-only models. As a result the CRASH model can be used to define a crush plane; one which is perpendicular to the impulse. In isolation the CRASH model produces a change in velocity as output. This data, or change in velocity data from any other source, can be used in the Neades and Smith model [**8**] to generate pre- and post-impact velocities identical to those produced by the Brach and Ishikawa models.

The momentum-only models and the CRASH model produce the same solutions when equivalent input data is used. Investigators can therefore choose any of the models for their analyses and that choice will depend amongst other considerations on the availability of data for a particular collision.

Acknowledgments
The author wishes to thank Dr Roy Smith of De Montfort University for his helpful comments and advice during the preparation of this article.

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Appendix A: Notation

d	distance of point of action from centre of mass
e	coefficient of restitution
E	energy absorbed by each vehicle
h	perpendicular distance from the vehicle's centre of mass to the line of action of $\boldsymbol{P}$
I	yaw moment of inertia
k	radius of gyration for each vehicle
m	mass of each vehicle
$\boldsymbol{n}$	unit vector perpendicular to $\boldsymbol{P}_1$
$\boldsymbol{p}$	unit vector in the direction of $\boldsymbol{P}_1$
$\boldsymbol{P}$	impulse due to the collision
$\boldsymbol{u}$	linear velocity of the centre of mass of each vehicle before impact
U	component of the velocity of the point of action before impact
$\boldsymbol{v}$	linear velocity of the centre of mass of each vehicle after impact
V	component of the velocity of the point of action after impact
W	work performed at point of action of impulse
γ	scalar factor $k^2/(k^2+h^2)$
Γ	angle of the impact (crush) plane to the x-y axes
δ	scalar factor $1+h^2/ k^2$ i.e. $1/\gamma$
λ	angle between closing velocity vector and direction of travel of vehicle
θ	principal direction of force
ϕ	angle of point of action relative to vehicle heading
ψ	angle of vehicle relative to the x-y axes (vehicle heading)
$\Delta\boldsymbol{v}$	velocity change at centre of mass due to impact, $\boldsymbol{v} - \boldsymbol{u}$
$\Delta\omega$	change in angular velocity due to the impact, $\Omega - \omega$
ω	angular velocity of the vehicle before impact
Ω	angular velocity of the vehicle after impact

Subscripts

n	motion normal to the impact plane
p	motion along the line of action of the impulse $\boldsymbol{P}$
pt	motion perpendicular or tangential to the impulse $\boldsymbol{P}$
t	motion perpendicular or tangential to the impact plane
1	vehicle 1
2	vehicle 2
R	relative value at the point of action of the impulse $\boldsymbol{P}$

Appendix B: Solution equations for Brach's planar impact model

Brach and Brach [**9**] show that the solution to this model can be expressed as

$$
\begin{aligned}
v_{1n} &= u_{1n} + \overline{m}(1+e_n)U_{Rn}q/m_1, \\
v_{1t} &= u_{1t} + \mu\overline{m}(1+e_n)U_{Rn}q/m_1, \\
v_{2n} &= u_{2n} - \overline{m}(1+e_n)U_{Rn}q/m_2, \\
v_{2t} &= u_{2t} - \mu\overline{m}(1+e_n)U_{Rn}q/m_2, \\
\Omega_1 &= \omega_1 + \overline{m}(1+e_n)U_{Rn}(h_1 - \mu h_{1t})q/(m_1k_1^2), \\
\Omega_2 &= \omega_2 + \overline{m}(1+e_n)U_{Rn}(h_2 - \mu h_{2t})q/(m_2k_2^2)
\end{aligned}
$$

where

$$
\begin{aligned}
&\overline{m} = m_1m_2/(m_1+m_2), \\
&e_n = -(V_{Rn}/U_{Rn}), \\
&\mu = P_t/P_n, \\
&U_{Rn} = u_{2n} - h_2\omega_2 - u_{1n} + h_1\omega_1, \qquad V_{Rn} = v_{2n} - h_2\Omega_2 - v_{1n} + h_1\Omega_1, \\
&U_{Rt} = u_{2t} - h_{2t}\omega_2 - u_{1t} + h_{1t}\omega_2, \qquad V_{Rt} = v_{2t} + h_{2t}\Omega_2 - v_{1t} + h_{1t}\Omega_1, \\
&\frac{1}{q} = 1 + \frac{\overline{m}h_1^2}{m_1k_1^2} + \frac{\overline{m}h_2^2}{m_2k_2^2} - \mu\left(\frac{\overline{m}h_1h_{1t}}{m_1k_1^2} + \frac{\overline{m}h_2h_{2t}}{m_2k_2^2}\right) = A - \mu B, \\
&h_1 = d_1\sin(\psi_1 + f_1 - \Gamma) \qquad h_{1t} = d_1\cos(\psi_1 + f_1 - \Gamma), \\
&h_2 = d_2\sin(\psi_2 + f_2 - \Gamma) \qquad h_{2t} = d_2\cos(\psi_2 + f_2 - \Gamma).
\end{aligned}
$$

The critical impulse ratio μ_0 is further defined as

$$
\mu_0 = \frac{rA + B(1+e_n)}{(1+e_n)(1+C) + rB}
$$

Where

$$
\begin{aligned}
r &= U_{Rt}/U_{Rn}, \\
A &= 1 + \frac{\overline{m}h_1^2}{m_1k_1^2} + \frac{\overline{m}h_2^2}{m_2k_2^2}, \\
B &= \frac{\overline{m}h_1h_{1t}}{m_1k_1^2} + \frac{\overline{m}h_2h_{2t}}{m_2k_2^2}, \\
C &= \frac{\overline{m}h_{1t}^2}{m_1k_1^2} + \frac{\overline{m}h_{2t}^2}{m_2k_2^2}.
\end{aligned}
$$

The subscripts *n* and *t* represent component variables normal and tangential to the impact or crush plane.

Appendix C: Solution equations for Ishikawa's model

Ishikawa [**6**] and [**19**] shows that the solution to this model can be expressed as

$$P_n = \frac{1}{(1-m_n m_t m_0^2)}\left[m_n U_{Rn}(1+e_n) + m_n m_t m_0 U_{Rt}(1+e_t)\right],$$
$$P_t = \frac{1}{(1-m_n m_t m_0^2)}\left[m_t U_{Rt}(1+e_t) + m_n m_t m_0 U_{Rn}(1+e_n)\right].$$

where e_n and e_t can be found from

$$V_{Rn} = -e_n U_{Rn}, \qquad V_{Rt} = -e_t U_{Rt}.$$

The relative speeds of the point of application of the impulse are defined as

$$U_{Rn} = u_{2n} - h_2\omega_2 - u_{1n} + h_1\omega_1,$$
$$V_{Rn} = v_{2n} - h_2\Omega_2 - v_{1n} + h_1\Omega_1,$$
$$U_{Rt} = u_{2t} + h_{2t}\omega_2 - u_{1t} - h_{1t}\omega_1,$$
$$V_{Rt} = v_{2t} + h_{2t}\Omega_2 - v_{1t} - h_{1t}\Omega_1.$$

The mass ratios used extensively by Ishikawa are defined as

$$m_n = \frac{\gamma_{1n} m_1 \gamma_{2n} m_2}{\gamma_{1n} m_1 + \gamma_{2n} m_2},$$
$$m_t = \frac{\gamma_{1t} m_1 \gamma_{2t} m_2}{\gamma_{1t} m_1 + \gamma_{2t} m_2},$$
$$m_0 = \frac{h_1 h_{1t}}{m_1 k_1^2} + \frac{h_2 h_{2t}}{m_2 k_2^2}.$$

where

$$\gamma_{1n} = \frac{k_1^2}{k_1^2 + h_1^2} \qquad \gamma_{2n} = \frac{k_2^2}{k_2^2 + h_2^2}$$
$$, $$
$$\gamma_{1t} = \frac{k_1^2}{k_1^2 + h_{1t}^2} \qquad \gamma_{2t} = \frac{k_2^2}{k_2^2 + h_{2t}^2}$$

$$e_t = \frac{m_n U_{Rn}(1+e_n)(\mu - m_t m_0)}{m_t U_{Rt}(1-\mu m_n m_0)} - 1,$$
$$\mu = \frac{P_t}{P_n}.$$

Appendix D: Conversion between Brach's and Ishikawa's models

The equivalence between the various parameters used by Brach and Brach [**9**] and the impact model by Ishikawa [**6**] are summarised below

$$m_n = \frac{\bar{m}}{A}, \qquad m_t = \frac{\bar{m}}{(1+C)}, \qquad m_0 = \frac{B}{\bar{m}}.$$

From these equivalences the following products and ratios can be derived

$$m_n m_0 = \frac{B}{A},$$

$$m_t m_0 = \frac{B}{(1+C)},$$

$$\frac{m_n}{m_t} = \frac{1+C}{A},$$

$$m_n m_t m_0^2 = \frac{B^2}{A(1+C)},$$

$$m_n m_t m_0 = \frac{\bar{m}B}{A(1+C)}.$$

Using Ishikawa's notation μ can be expressed as

$$\mu = \frac{P_t}{P_n} = \frac{m_n m_t m_0 U_{Rn}(1+e_n) + m_t U_{Rt}(1+e_t)}{m_n m_t m_0 U_{Rt}(1+e_t) + m_n U_{Rn}(1+e_n)}.$$

Using the conversion factors specified above, μ can be expressed using Brach's notation and the tangential coefficient of restitution e_t as

$$\mu = \frac{(1+e_t)rA + B(1+e_n)}{(1+e_n)(1+C) + rB(1+e_t)}.$$

Using the same notation, e_t can be expressed in terms of μ as

$$e_t = \frac{(1+e_n)[\mu(1+C) - B]}{r(A - \mu B)} - 1.$$

Appendix E: Tables

Table 1: RICSAC Test 9 Data

Parameter	**Vehicle 1**	**Vehicle 2**
m (kg)	1023.0	2221.2
I (kg m^2)	1323.3	5359.6
d (m)	1.463	1.707
ψ (°)	0	90
ϕ (°)	6.0	-29.7
U (ms^{-1})	-9.48	9.48

Table 2: RICSAC Test 9 Results

Parameter	**Vehicle 1**			**Vehicle 2**		
	Actual	**Brach** [**9**]	**Neades**	**Actual**	**Brach** [**9**]	**Neades**
V_x (ms^{-1})	-1.55	-1.89	-1.52	-2.23	-3.51	-3.07
V_y (ms^{-1})	5.36	3.90	3.85	6.16	7.68	6.26
V (ms^{-1})	5.58	4.32	4.14	6.55	8.44	6.97
Exit angle (°)	73.9	64.4	68.5	70.1	65.5	63.9
Ω (°/s)	-180	-200	-203	45	87	72
Δv (ms^{-1})	8.72$^{(1)}$	8.55	7.69	3.84$^{(1)}$	3.94	3.54
PDOF θ (°)	-30$^{(2)}$	-27.1	-30	60$^{(2)}$	62.9	60

[1] determined from accelerometer data Brach and Smith [**22**]
[2] visually estimated value from Jones and Baum [**20**]